

ON THE GENERIC VANISHING THEOREM OF CARTIER MODULES

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ABSTRACT. We generalize the Generic Vanishing theorem by Hacon and Patakfalvi in the spirit of Pareschi and Popa. We give several examples illustrating the pathologies appearing in the positive characteristic setting.

1. INTRODUCTION

Let X be a smooth projective variety, let $a : X \rightarrow A$ be the Albanese morphism and let

$$V^i(\omega_X) = \{P \in \text{Pic}^0(X) \mid H^i(X, \omega_X \otimes P) \neq 0\}$$

be the cohomology support loci. In [GL90] and [GL91], Green and Lazarsfeld proved the following theorem which is an essential result in the study of irregular varieties (see, for example, [Fujino09], [JLT11] and [Simpson93]).

Theorem 1.1. *Let X be a smooth complex projective variety. Then every irreducible component of $V^i(\omega_X)$ is a translate of a subtorus of $\text{Pic}^0(X)$ of codimension at least*

$$i - \dim X + \dim a(X).$$

If X has maximal Albanese dimension, then there are inclusions:

$$V^0(\omega_X) \supset V^1(\omega_X) \supset \cdots \supset V^{\dim X}(\omega_X) = \{\mathcal{O}_X\}.$$

The theorem was first proven using Hodge theory. An alternative point of view using the Fourier-Mukai transforms $R\hat{S}$ and RS emerged in [Hacon04] and [PP11]. Specifically, in [PP11], Pareschi and Popa proved the following theorem.

Theorem 1.2. *Let A be an abelian variety. Let \mathcal{F} be a coherent sheaf on A . The following are equivalent:*

- (1) *For any sufficiently ample line bundle L on \hat{A} , $H^i(A, \mathcal{F} \otimes \hat{L}^\vee) = 0$ for any $i > 0$, where $\hat{L} = R^0\hat{S}(L) = R\hat{S}(L)$,*
- (2) *$R^i\hat{S}(D_A(\mathcal{F})) = 0$ for any $i \neq 0$,*
- (3) *$\text{codim Supp } R^i\hat{S}(\mathcal{F}) \geq i$ for any $i \geq 0$, and*
- (4) *$\text{codim } V^i(\mathcal{F}) \geq i$ for any $i \geq 0$.*

The theorem holds even in positive characteristic. But, in order to apply it to the canonical bundle of irregular varieties via Albanese maps, we need the result of Kollár in [Kollar86I] and [Kollar86II] or Grauert-Reimanschneider Vanishing which is known to fail in positive characteristic (see [HK12]). In [HP13], Hacon and Patakfalvi suggested that, instead of $R^i a_* \omega_X$, we should consider the inverse limit

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of the push-forwards of $R^i a_* \omega_X$ by the Frobenius map. In particular, they proved the the following results.

Theorem 1.3. *Let k be an algebraically closed field of characteristic $p > 0$ and A be an abelian variety over k . Let $\{\Omega_e\}$ be an inverse system of coherent sheaves on A such that for any sufficiently ample line bundle L on \hat{A} and any $e \gg 0$, $H^i(A, \Omega_e \otimes \hat{L}^\vee) = 0$ for all $i > 0$. Then, the complex*

$$\Lambda = \varinjlim R\hat{S}(D_A(\Omega_e))$$

is a quasi-coherent sheaf in degree 0, i.e., $\Lambda = \mathcal{H}^0(\Lambda)$. Here \varinjlim is a generalization of direct limit to derived category (see Section 2.1).

Theorem 1.4. *If $\{\Omega_e\}$ is a Cartier module, then it satisfies the condition in Theorem 1.3. In particular, let X be a normal, projective variety over an algebraically closed field k of characteristic $p > 0$, then $\Omega_e = F_*^e S^0 a_* \omega_X$ satisfies the condition in Theorem 1.3.*

One should regard Theorem 1.3 as a generalization of (1) \Rightarrow (2) in Theorem 1.2. It is a natural question to ask what is the appropriate generalization of the statements for (3) and (4) in Theorem 1.2 to the positive characteristic setting and if all the resulting conditions are equivalent to each other.

In this paper, we generalize Hacon and Patakfalvi's theorem as follows.

Theorem 1.5. *(See Theorem 4.1, 4.2 and 4.4) Let A be an abelian variety. Let $\{\Omega_e\}$ be an inverse system of coherent sheaves on A satisfying the Mittag-Leffler condition and let $\Omega = \varprojlim \Omega_e$. Let $\Lambda_e = R\hat{S}(D_A(\Omega_e))$ and $\Lambda = \varinjlim \Lambda_e$. The following are equivalent:*

- (1) *For any ample line bundle L on \hat{A} , $H^i(A, \Omega \otimes \hat{L}^\vee) = 0$ for any $i > 0$.*
- (1') *For any fixed positive integer e and any $i > 0$, the homomorphism*

$$H^i(A, \Omega \otimes \hat{L}^\vee) \rightarrow H^i(A, \Omega_e \otimes \hat{L}^\vee)$$

is 0 for any sufficiently ample line bundle L .

- (2) $\mathcal{H}^i(\Lambda) = 0$ for any $i \neq 0$.

If any of these conditions is satisfied, then we will call $\{\Omega_e\}$ a GV-inverse system of coherent sheaves.

These conditions imply the following equivalent conditions:

- (3) *For any scheme-theoretic point $P \in A$, if $\dim P > i$, then P is not in the support of*

$$\mathrm{im}(R^i \hat{S}(\Omega) \rightarrow R^i \hat{S}(\Omega_e))$$

for any e .

- (3') *For any scheme-theoretic point $P \in A$, if $\dim P > i$, then P is not in the support of*

$$\mathrm{im}(\varprojlim R^i \hat{S}(\Omega_e) \rightarrow R^i \hat{S}(\Omega_e))$$

for any e .

If $\{R^i \hat{S}(\Omega_e)\}$ satisfies the Mittag-Leffler condition for any $i \geq 0$, then (3) and (3') also imply (1), (1') and (2).

We should make a remark that even if $\{\Omega_e\}$ is a Cartier module, $\{R^i \hat{S}(\Omega_e)\}$ does not necessarily satisfy the Mittag-Leffler condition (see Example 3.3). We are unable to prove the equivalence in this case. On the other hand, the statement

about $V^i(\Omega)$ is still missing. We will give an example (see Example 3.4) where the chain of inclusions for $V^i(\Omega)$ fails. Since the support of $\text{im}(R^i\hat{S}(\Omega) \rightarrow R^i\hat{S}(\Omega_e))$ is usually not closed (see Example 3.2), it is not a good idea to talk about its codimension.

In a sequence of papers [PP03, PP04, PP08], Pareschi and Popa introduced the notion of M-regularity which parallels and strengthens the usual Castelnuovo-Mumford regularity with respect to polarizations on abelian varieties and developed several results on global generation. In [PP08], the following characterization of M-regularity is given.

Theorem 1.6. *Let A be an abelian variety and \mathcal{F} be a coherent sheaf on A satisfying the Generic Vanishing conditions. The following conditions are equivalent:*

- (1) \mathcal{F} is M-regular, i.e., $R^0\hat{S}(D_A(\mathcal{F}))$ is torsion-free.
- (2) $\text{codim Supp } R^i\hat{S}(\mathcal{F}) > i$ for any $i \geq 0$
- (3) $\text{codim } V^i(\mathcal{F}) > i$ for any $i \geq 0$.

We will generalize the theorem above to inverse systems as follows.

Theorem 1.7. *(See Theorem 4.2) Let A be an abelian variety and $\{\Omega_e\}$ be a GV-inverse system of coherent sheaves on A such that*

- (1) $\{\Omega_e\}$ is M-regular in the sense that $\mathcal{H}^0(\Lambda)$ is torsion-free.

Then

- (2) *for any scheme-theoretic point $P \in A$, if $\dim P \geq i$, then P is not in the support of*

$$\text{im}(R^i\hat{S}(\Omega) \rightarrow R^i\hat{S}(\Omega_e))$$

for any e .

We are unable to prove the converse statement of the above theorem.

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2. PRELIMINARIES

We work over a perfect field k of arbitrary characteristic.

2.1. Derived category. We recall some basic notations in derived category. For details, we refer to [HP13, Section 2.1] and [Neeman96].

Given a variety X , let $D(X)$ be the derived category of \mathcal{O}_X -modules and $D_{qc}(X)$ (resp. $D_c(X)$) be the full subcategory consisting of bounded complex whose cohomologies are quasi-coherent (resp. coherent). For any object $\mathcal{E} \in D_{qc}(X)$, $\mathcal{E}[n]$ denotes the object obtained by shifting \mathcal{E} , n places to the left, and $\mathcal{H}^n(\mathcal{E})$ denotes the n -th homology of a complex representing \mathcal{E} .

Let X be a variety of dimension n and $\omega_X^\bullet = p^!\mathcal{O}_k$ denote its dualizing complex such that $\mathcal{H}^{-n}(\omega_X^\bullet) \cong \omega_X$. The dualizing functor D_X is defined by $D_X(\mathcal{E}) = R\mathcal{H}om(\mathcal{E}, \omega_X^\bullet)$ for any $\mathcal{E} \in D_{qc}(X)$. We have Grothendieck Duality:

Theorem 2.1 (Grothendieck Duality). *Let $f : X \rightarrow Y$ be a proper morphism of quasi-projective varieties over a field k . Then*

$$Rf_*(D_X(\mathcal{E})) = D_Y(Rf_*(\mathcal{E}))$$

for any $\mathcal{E} \in D_{qc}(X)$.

As a generalization of direct limit in triangulated category, the homotopy colimit is defined as follows.

Definition 2.2. Let $\{\mathcal{C}_e\}$ be a direct system of objects in $D_{qc}(X)$,

$$\mathcal{C}_1 \xrightarrow{f_1} \mathcal{C}_2 \xrightarrow{f_2} \dots$$

The homotopy colimit $\operatorname{hocolim} \mathcal{C}_e$ is defined by the following triangle

$$\bigoplus \mathcal{C}_e \xrightarrow{\operatorname{id} - \bigoplus f_e} \bigoplus \mathcal{C}_e \rightarrow \operatorname{hocolim} \mathcal{C}_e \rightarrow \bigoplus \mathcal{C}_e[1].$$

Lemma 2.3. *Homotopy colimits commute with tensor products, pullbacks and pushforwards. In particular, we have*

- (1) $\operatorname{hocolim} \mathcal{H}^i(\mathcal{C}_e) = \mathcal{H}^i(\operatorname{hocolim} \mathcal{C}_e)$, and
- (2) $\operatorname{hocolim} R^i\Gamma(\mathcal{C}_e) = R^i\Gamma(\operatorname{hocolim} \mathcal{C}_e)$

Similarly, the homotopy limit is defined as:

Definition 2.4. Let $\{\mathcal{C}_e\}$ be an inverse system of objects in $D_{qc}(X)$,

$$\mathcal{C}_1 \xleftarrow{f_1} \mathcal{C}_2 \xleftarrow{f_2} \dots$$

The homotopy limit $\operatorname{holim} \mathcal{C}_e$ is defined by the following triangle

$$\operatorname{holim} \mathcal{C}_e \rightarrow \prod \mathcal{C}_e \xrightarrow{\operatorname{id} - \prod f_e} \prod \mathcal{C}_e \rightarrow \operatorname{holim} \mathcal{C}_e[1].$$

If \mathcal{C}_e are coherent sheaves, then $\operatorname{hocolim} \mathcal{C}_i = \varinjlim \mathcal{C}_e$.

Lemma 2.5. *If $\{\mathcal{C}_e\}$ is a direct system in $D_{qc}(X)$ and $\mathcal{D} \in D_{qc}(X)$, then*

$$R\mathcal{H}om(\operatorname{hocolim} \mathcal{C}_e, \mathcal{D}) \cong \operatorname{holim} R\mathcal{H}om(\mathcal{C}_e, \mathcal{D}).$$

In particular,

$$D_X(\operatorname{hocolim} \mathcal{C}_e) \cong \operatorname{holim} D_X(\mathcal{C}_e).$$

2.2. Fourier-Mukai transform. Let \hat{A} be the dual abelian variety of A . Let P be the normalized Poincare line bundle on $A \times \hat{A}$. Let p_A and $p_{\hat{A}}$ be the projection from $A \times \hat{A}$ to A and \hat{A} , respectively. Let \hat{S} be the functor between \mathcal{O}_A -modules and $\mathcal{O}_{\hat{A}}$ -modules defined as:

$$\hat{S}(\mathcal{F}) = p_{A,*}(p_A^*\mathcal{F} \otimes P).$$

The **Fourier-Mukai transform** $R\hat{S} : D(A) \rightarrow D(\hat{A})$ is the right derived functor of \hat{S} . Similarly, we define $RS : D(\hat{A}) \rightarrow D(A)$ as the right derived functor of $S(\mathcal{G}) = p_{A,*}(p_{\hat{A}}^*\mathcal{G} \otimes P)$. We recall the following propositions from [Mukai81] and [HP13].

Proposition 2.6. (See [Mukai81, Theorem 2.2][HP13, Theorem 2.18]) *The following properties hold on $D_{qc}(A)$ and $D_{qc}(\hat{A})$.*

$$RS \circ R\hat{S} = (-1_A)^*[-g] \quad R\hat{S} \circ RS = (-1_{\hat{A}})^*[-g],$$

where -1_A is the inverse on A and $[-g]$ denotes the shift by g places to the right.

Proposition 2.7. (See [Mukai81, Corollary 2.5]) *For all objects $\mathcal{E}, \mathcal{E}' \in D_{qc}(A)$,*

$$\mathrm{Hom}_{D_{qc}(A)}(\mathcal{E}, \mathcal{E}') \cong \mathrm{Hom}_{D_{qc}(\hat{A})}(R\hat{S}(\mathcal{E}), R\hat{S}(\mathcal{E}')).$$

Proposition 2.8. (See [Mukai81, 3.8][HP13, Lemma 2.20]) *We have $D_A \circ RS = (-1_A)^*(R\hat{S} \circ D_{\hat{A}})[g]$ on $D_{qc}(A)$.*

The Fourier-Mukai transform commutes with homotopical colimit.

Proposition 2.9. (See [HP13, Lemma 2.23]) *Let $\{\Lambda_e\}$ be a direct system in $D_{qc}(A)$. Then $R\hat{S}(\varinjlim \Lambda_e) = \varinjlim R\hat{S}(\Lambda_e)$.*

The Fourier-Mukai transform exchanges direct and inverse images of isogenies.

Proposition 2.10. (See [Mukai81, 3.4][HP13, Lemma 2.22]) *Let $\phi : A \rightarrow B$ be an isogeny of abelian varieties and $\hat{\phi} : \hat{B} \rightarrow \hat{A}$ be the dual isogeny. The following equalities hold on $D_{qc}(B)$ and $D_{qc}(A)$:*

$$\phi^* \circ RS_B \cong RS_A \circ \hat{\phi}_*,$$

$$\phi_* \circ RS_A \cong RS_B \circ \hat{\phi}^*.$$

We will use the following consequence of the projection formula:

Proposition 2.11. (See [PP11, Lemma 2.1]) *For all objects $\mathcal{E} \in D_c(A)$ and $\mathcal{E}' \in D_c(\hat{A})$,*

$$H^i(A, \mathcal{E} \otimes RS(\mathcal{E}')) = H^i(\hat{A}, R\hat{S}(\mathcal{E}) \otimes \mathcal{E}').$$

2.3. Inverse limit. We refer to [Hartshorne75, Chapter I, §4] and [EGA III, 0_{III}, §13] for details in this section.

Let $\{\Omega_e\}$ be an inverse system of coherent sheaves. We say $\{\Omega_e\}$ satisfies **the Mittag-Leffler condition**, if for any $e \geq 0$ the image of $\Omega_{e'} \rightarrow \Omega_e$ stabilized for e' sufficiently large. The inverse limit functor is always left exact in the sense that if $\{\mathcal{F}_e\}$, $\{\mathcal{G}_e\}$ and $\{\mathcal{H}_e\}$ are inverse systems of coherent sheaves and the following exact sequences

$$0 \rightarrow \mathcal{F}_e \rightarrow \mathcal{G}_e \rightarrow \mathcal{H}_e \rightarrow 0$$

are compatible with maps in the inverse systems, then

$$0 \rightarrow \varprojlim \mathcal{F}_e \rightarrow \varprojlim \mathcal{G}_e \rightarrow \varprojlim \mathcal{H}_e$$

is exact in the category of quasi-coherent sheaves. By a theorem of Roos [Roos61], the right derived functors $R^i \varprojlim = 0$ for $i \geq 2$. Hence, we have a long exact sequence

$$0 \rightarrow \varprojlim \mathcal{F}_e \rightarrow \varprojlim \mathcal{G}_e \rightarrow \varprojlim \mathcal{H}_e \rightarrow R^1 \varprojlim \mathcal{F}_e \rightarrow R^1 \varprojlim \mathcal{G}_e \rightarrow R^1 \varprojlim \mathcal{H}_e \rightarrow 0.$$

Lemma 2.12. (See [Hartshorne75, Corollary I.4.3]) *If $\{\Omega_e\}$ satisfies the Mittag-Leffler condition, then $R^1 \varprojlim \Omega_e = 0$.*

Theorem 2.13. (See [Hartshorne75, Theorem I.4.5]) *Let $\{\Omega_e\}$ be an inverse system of coherent sheaves on a variety X . Let T be a functor on $D(X)$ which commutes with arbitrary direct products. Suppose that $\{\Omega_e\}$ satisfies the Mittag-Leffler condition. Then for each i , there is an exact sequence*

$$0 \rightarrow R^1 \varprojlim R^{i-1}T(\Omega_e) \rightarrow R^iT(\varprojlim \Omega_e) \rightarrow \varprojlim R^iT(\Omega_e) \rightarrow 0.$$

In particular, if for some i , $\{R^{i-1}T(\Omega_e)\}$ satisfies the Mittag-Leffler condition, then $R^iT(\varprojlim \Omega_e) \cong \varprojlim R^iT(\Omega_e)$.

In applications, the functor T above can be Γ , f_* , S and \hat{S} as in the Fourier-Mukai transform.

2.4. Spectral sequence. We recall the definition of spectral sequence from [EGA III, 0_{III}, §11]. We also refer to [GM03, III.7]. Let \mathcal{C} be an abelian category. A **(biregular) spectral sequence** E on \mathcal{C} consists of the following ingredients:

- (1) A family of objects $\{E_r^{p,q}\}$ in \mathcal{C} , where $p, q, r \in \mathbb{Z}$ and $r \geq 2$, such that for any fixed pair (p, q) , $E_r^{p,q}$ stabilizes when r is sufficiently large. We denote the stable objects by $E_\infty^{p,q}$.
- (2) A family of morphisms $d_r^{p,q} : E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$ satisfying
$$d_r^{p+r, q-r+1} \circ d_r^{p,q} = 0.$$
- (3) A family of isomorphisms $\alpha_r^{p,q} : \ker(d_r^{p,q}) / \text{im}(d_r^{p-r, q+r-1}) \xrightarrow{\sim} E_{r+1}^{p,q}$.
- (4) A family of objects $\{E^n\}$ in \mathcal{C} . For every E_n , there is a bounded decreasing filtration $\{F^p E^n\}$ in the sense that there is some p such that $F^p E^n = E^n$ and there is some p such that $F^p E^n = 0$.
- (5) A family of isomorphisms $\beta^{p,q} : E_\infty^{p,q} \xrightarrow{\sim} F^p E^{p+q} / F^{p+1} E^{p+q}$.

We say the spectral sequence $\{E_r^{p,q}\}$ converges to $\{E^n\}$ and write

$$E_2^{p,q} \Rightarrow E^{p+q}.$$

A morphism $\phi : E \rightarrow H$ between two spectral sequences on \mathcal{C} is a family of morphisms $\phi_r^{p,q} : E_r^{p,q} \rightarrow H_r^{p,q}$ and $\phi^n : E^n \rightarrow H^n$ such that ϕ is compatible with d , α , the filtration and β .

Theorem 2.14 (Grothendieck). *Let \mathcal{A}, \mathcal{B} and \mathcal{C} be abelian categories. Let $F : \mathcal{A} \rightarrow \mathcal{B}$ and $G : \mathcal{B} \rightarrow \mathcal{C}$ be two left exact functors. Suppose every object in \mathcal{A} and \mathcal{B} has finite injective resolution and the class of injective objects in \mathcal{B} is sufficiently large. Then for any object X in \mathcal{A} , there exists a spectral sequence with $E_2^{p,q} = R^p G(R^q F(X))$ converging to $E^n = R^n(G \circ F)(X)$. It is functorial in X .*

In this paper, we need to know when the morphisms between the limits are zero.

Lemma 2.15. *Let*

$$\begin{array}{ccc} E_2^{i,j} & \Longrightarrow & E^{i+j} \\ \downarrow \phi_2^{i,j} & & \downarrow \phi^{i+j} \\ H_2^{i,j} & \Longrightarrow & H^{i+j} \end{array}$$

be two spectral sequences with commutative maps. Let l and a be integers. Suppose that $E_2^{i, l-i} = 0$ for $i < a$, $H_2^{i, l-i} = 0$ for $i > a$ and $\phi_2^{a, l-a} = 0$. Then $\phi^l = 0$.

Proof. Since

$$E_3^{i,j} \cong \ker(E_2^{i,j} \rightarrow E_2^{i+2,j-1}) / \operatorname{im}(E_2^{i-2,j+1} \rightarrow E_2^{i,j}),$$

it follows that $E_3^{i,l-i} = 0$ for $i < a$, $H_3^{i,l-i} = 0$ for $i > a$ and $\phi_3^{a,l-a} = 0$. Hence $E_\infty^{i,l-i} = 0$ for $i < a$, $H_\infty^{i,l-i} = 0$ for $i > a$ and $\phi_\infty^{a,l-a} = 0$, by induction. Let $\{F^p E^l\}$ and $\{F^p H^l\}$ be the filtration for E^l and H^l , respectively.

We prove by induction that $F^p \phi^l : F^p E^l \rightarrow F^p H^l$ is zero for any p . For $p \geq a+1$, we have that $F^p H^l = 0$. Hence $F^p \phi^l = 0$. Suppose $F^{i+1} \phi^l = 0$ for some $i \leq a$. Since we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & F^{i+1} E^l & \longrightarrow & F^i E^l & \longrightarrow & E_\infty^{i,l-i} \longrightarrow 0 \\ & & \downarrow F^{i+1} \phi^l & & \downarrow F^i \phi^l & & \downarrow \phi_\infty^{i,l-i} \\ 0 & \longrightarrow & F^{i+1} H^l & \longrightarrow & F^i H^l & \longrightarrow & H_\infty^{i,l-i} \longrightarrow 0 \end{array}$$

- (1) If $i = a$, then $F^{i+1} H^l = 0$. By the Snake Lemma, we have that

$$\operatorname{coker} F^i \phi^l \cong \operatorname{coker} \phi_\infty^{i,l-i} \cong H_\infty^{i,l-i} \cong F^i H^l.$$

Hence, $F^i \phi^l = 0$.

- (2) If $i < a$, then $E_\infty^{i,l-i} = 0$. By induction, we may assume that $F^{i+1} \phi^l = 0$. By the Snake Lemma, we have that

$$\ker F^i \phi^l \cong \ker F^{i+1} \phi^l \cong F^{i+1} E^l \cong F^i E^l.$$

We also obtain that $F^i \phi^l = 0$.

The lemma follows. \square

2.5. Frobenius morphism and Cartier module. Let k be a perfect field of positive characteristic. Let X be a normal variety over k . The (absolute) Frobenius morphism $F : X \rightarrow X$ is defined as identity on the topological space and taking p -th power on local sections. We denote by F^e the e -th iteration of F .

A **Cartier module** is a coherent sheaf \mathcal{F} on X equipped with an \mathcal{O}_X -linear map

$$\phi : F_*^e \mathcal{F} \rightarrow \mathcal{F},$$

which is also called a p^{-e} -linear map in [BS13]. A well-known example of Cartier module is the canonical sheaf ω_X with the trace map

$$\operatorname{Tr} : F_* \omega_X \rightarrow \omega_X,$$

which is the dual of the structure map $\mathcal{O}_X \rightarrow F_* \mathcal{O}_X$.

Suppose (\mathcal{F}, ϕ) is a Cartier module on X . It is easy to see that we can iterate ϕ to get a sequence of maps

$$\dots \rightarrow F_*^{3e} \mathcal{F} \xrightarrow{F_*^{2e} \phi} F_*^{2e} \mathcal{F} \xrightarrow{F_*^e \phi} F_*^e \mathcal{F} \xrightarrow{\phi} \mathcal{F}.$$

It is known that this inverse system of coherent sheafs satisfies the Mittag-Leffler condition [BS13, Proposition 8.1.4].

3. EXAMPLES

We will propose several examples for the pathologies appearing in the context of Cartier modules.

Example 3.1. This example first appears in [HP13, Example 3.21]. Let A be an elliptic curve, $\Omega_0 = \omega_A$, $\Omega_e = F_*^e \Omega_0$ and $\alpha : F_* \Omega_0 \rightarrow \Omega_0$ be the trace map.

When A is ordinary, then $\Lambda = \varinjlim R\hat{S}(D_A(\Omega_e)) = \bigoplus_{y \in A_p} k(y)$ where A_p denotes the set of all p^∞ -torsion points in \hat{A} and $R\hat{S}(\Omega) = \prod_{y \in A_p} k(y)[-1]$. Hence

$$\text{Supp } R^1\hat{S}(\Omega) = A_p,$$

which is a countable dense set by [MvdG, 5.30]. By [HP13, Proposition 3.18], $V^1(\Omega) = A_p$ which is dense in \hat{A} .

However, suppose \mathcal{F} is a coherent sheaf satisfying the Generic Vanishing conditions, then $V^1(\mathcal{F})$ is a closed subset of dimension 0 or empty by Theorem 1.2.

We should notice that in this example, although $\text{Supp } R^1\hat{S}(\Omega)$ is not closed, the support of the image of $R^1\hat{S}(\Omega) \rightarrow R^1\hat{S}(\Omega_e)$ is the set of p^e -torsion points which is closed for any $e > 0$. \square

It should be noticed that in the previous example, $\{R^i\hat{S}(\Omega_e)\}$ satisfies the Mittag-Leffler condition for any i . We will see that this is not valid in general in the following examples.

Example 3.2. Let A be an elliptic curve. Let $\hat{0} \in \hat{A}$ correspond to the trivial line bundle on A . Let $W_e = \mathcal{O}_{\hat{A}}(-e \cdot \hat{0})$ and $\psi_e : W_{e+1} \rightarrow W_e$ be the inclusion. Clearly, the inverse system of coherent sheaves $\{W_e\}$ does not satisfy the Mittag-Leffler condition. Since the W_e are antiample, $R^0S(W_e) = 0$. Let $\Omega_e = R^1S(W_e) = RS(W_e)[1]$. Notice that we have short exact sequences

$$0 \rightarrow W_{e+1} \rightarrow W_e \rightarrow k(\hat{0}) \rightarrow 0.$$

The Fourier-Mukai transform induces the following long exact sequences

$$\begin{array}{ccccccccc} R^0S(W_e) & \longrightarrow & R^0S(k(\hat{0})) & \longrightarrow & R^1S(W_{e+1}) & \longrightarrow & R^1S(W_e) & \longrightarrow & R^1S(k(\hat{0})) \\ \parallel & & \parallel & & \parallel & & \parallel & & \parallel \\ 0 & & \mathcal{O}_A & & \Omega_{e+1} & & \Omega_e & & 0. \end{array}$$

In particular, $\Omega_{e+1} \rightarrow \Omega_e$ is surjective. Thus $\{\Omega_e\}$ satisfies the Mittag-Leffler condition. On the other hand, $R^0\hat{S}(\Omega_e) = (-1_{\hat{A}})^*W_e$ does not satisfy the Mittag-Leffler condition.

We claim that $\{\Omega_e\}$ above is a GV-inverse system of coherent sheaves. Indeed,

$$R\hat{S}(D_A(\Omega_e)) = (-1_{\hat{A}})^*D_{\hat{A}}(R\hat{S}(\Omega_e))[-1] = D_{\hat{A}}(W_e)[-1] = \mathcal{O}_{\hat{A}}(e \cdot \hat{0}).$$

In particular, $R^{-1}\hat{S}(D_A(\Omega_e)) = 0$. Taking the direct limit, we conclude that $\mathcal{H}^{-1}(\Lambda) = 0$.

We first calculate the codimensions of the supports. Since $R^1\hat{S}(\Omega_e) = 0$ for any e , it is clear that $\varinjlim R^1\hat{S}(\Omega_e) = 0$. Since $\{\Omega_e\}$ satisfies the Mittag-Leffler condition, by Theorem 2.13, we have the following exact sequence

$$0 \rightarrow R^1\varinjlim R^0\hat{S}(\Omega_e) \rightarrow R^1\hat{S}(\Omega) \rightarrow \varinjlim R^1\hat{S}(\Omega_e) \rightarrow 0.$$

Thus $R^1\hat{S}(\Omega) \cong R^1\varprojlim R^0\hat{S}(\Omega_e) \cong (-1_A)^* R^1\varprojlim W_e$. Consider the following exact sequence of inverse systems

$$0 \rightarrow W_e \rightarrow \mathcal{O}_{\hat{A}} \rightarrow k[t]/(t^e) \rightarrow 0.$$

We have long exact sequence

$$0 \rightarrow \varprojlim W_e \rightarrow \mathcal{O}_{\hat{A}} \rightarrow \varprojlim k[t]/(t^e) \rightarrow R^1\varprojlim W_e \rightarrow R^1\varprojlim \mathcal{O}_{\hat{A}} = 0,$$

where the last equation follows by Lemma 2.12. We conclude that $R^1\varprojlim W_e$ is the skyscraper sheaf $k[[t]]/k[t]$ at $\hat{0}$. Hence

$$\text{Supp } R^1\hat{S}(\Omega) = \{\hat{0}\}$$

and

$$\text{Supp}(\text{im } R^1\hat{S}(\Omega) \rightarrow R^1\hat{S}(\Omega_e)) = \emptyset.$$

By Theorem 2.13, it is clear that $R^0\hat{S}(\Omega) = \varprojlim R^0\hat{S}(\Omega_e) = (-1)^*\varprojlim W_e$. Hence

$$\text{Supp } R^0\hat{S}(\Omega) = \hat{A} - \{\hat{0}\}$$

and

$$\text{Supp}(\text{im } R^0\hat{S}(\Omega) \rightarrow R^0\hat{S}(\Omega_e)) = \hat{A} - \{\hat{0}\}.$$

We now calculate the cohomology support loci. Let $\alpha \in \hat{A}$ and $P_\alpha \in \text{Pic}^0(A)$ be the corresponding topologically trivial line bundle. We have

$$\begin{aligned} H^i(A, \Omega_e \otimes P_\alpha) &\cong H^i(A, R^1S(W_e) \otimes P_\alpha) \\ &\cong H^{i+1}(A, RS(W_e) \otimes P_\alpha) \\ &\cong H^{i+1}(\hat{A}, W_e \otimes R\hat{S}(P_\alpha)) \\ &\cong H^i(\hat{A}, W_e \otimes R^1\hat{S}(P_\alpha)) \\ &\cong H^i(\hat{A}, W_e \otimes k(-\alpha)), \end{aligned}$$

where the third isomorphism is by Proposition 2.11. Hence $H^0(A, \Omega_e \otimes P_\alpha) \cong k$ and $H^1(A, \Omega_e \otimes P_\alpha) = 0$. Taking the inverse limit, we have $H^i(A, \Omega \otimes P_\alpha) = \varprojlim H^i(A, \Omega_e \otimes P_\alpha) = k(-\alpha)$ if $i = 0$ and $\alpha \neq \hat{0}$, and $H^i(A, \Omega \otimes P_\alpha) = 0$ otherwise. We conclude that

$$V^1(\Omega) = \emptyset$$

and

$$V^0(\Omega) = \hat{A} - \{\hat{0}\}.$$

In particular,

$$V^0(\Omega) = \text{Supp } R^0\hat{S}(\Omega) = \text{Supp}(\text{im } R^0\hat{S}(\Omega) \rightarrow R^0\hat{S}(\Omega_e)) = \hat{A} - \{\hat{0}\}$$

are not countable unions of closed subsets.

We should notice that $V^1(\Omega) \not\supseteq \text{Supp } R^1\hat{S}(\Omega)$. If \mathcal{F} is a coherent sheaf, then it is a consequence of cohomology and base change that $V^i(\mathcal{F}) \supseteq \text{Supp } R^i\hat{S}(\mathcal{F})$. \square

We can easily modify Example 3.2 to obtain a Cartier module.

Example 3.3. Let A be a supersingular elliptic curve. Let $\{W_e\}$ and $\{\Omega_e\}$ be the same as in Example 3.2. Let us consider the inverse system $\{\Omega_{p^e}\}$. Notice that since A is supersingular,

$$W_{p^e} = \mathcal{O}_{\hat{A}}(-p^e \cdot \hat{0}) = V^{*,e}(W_1),$$

where the Verschiebung $V : \hat{A} \rightarrow \hat{A}$ is the dual of the Frobenius. We have

$$\Omega_{p^e} \cong RS(W_{p^e})[1] \cong RS(V^{*,e}(W_1))[1] \cong F_*^e(RS(W_1))[1] \cong F_*^e\Omega_1,$$

where the third isomorphism is by Proposition 2.10. Thus, $\{\Omega_{p^e}\}$ is a Cartier module. The calculation of inverse limits remains unchanged as in Example 3.2. In particular, $V^0(\Omega) = \hat{A} - \{\hat{0}\}$ is not a countable union of closed subvarieties. This gives a negative answer to [HP13, Question 3.20]. \square

The following example shows that the chain of inclusions fails for GV-inverse system of coherent sheaves.

Example 3.4. Let A be an elliptic curve. Let $\Omega_0 = \mathcal{O}_A$ and Ω_{e+1} be the non-splitting extension of \mathcal{O}_A and Ω_e ,

$$0 \rightarrow \mathcal{O}_A \rightarrow \Omega_{e+1} \rightarrow \Omega_e \rightarrow 0.$$

Then $H^0(A, \Omega_e) \cong H^1(A, \Omega_e) \cong k$ for any $e \geq 0$. Since $\Omega_{e+1} \rightarrow \Omega_e$ is surjective, the inverse system $\{\Omega_e\}$ satisfies the Mittag-Leffler condition. It is easy to check that

$$R^{-1}\hat{S}(D_A(\Omega_e)) = 0$$

for any $e \geq 0$ by induction. Hence, $\{\Omega_e\}$ is a GV-inverse system.

We now compute the cohomology support loci $V^0(\Omega)$ and $V^1(\Omega)$. Suppose $P_\alpha \in \text{Pic}^0(A)$ and $P_\alpha \neq \mathcal{O}_A$. By the long exact sequence,

$$\begin{aligned} 0 \rightarrow H^0(A, P_\alpha) \rightarrow H^0(A, \Omega_{e+1} \otimes P_\alpha) \rightarrow H^0(A, \Omega_e \otimes P_\alpha) \\ \rightarrow H^1(A, P_\alpha) \rightarrow H^1(A, \Omega_{e+1} \otimes P_\alpha) \rightarrow H^1(A, \Omega_e \otimes P_\alpha) \rightarrow 0, \end{aligned}$$

we have that

$$H^0(A, \Omega_e \otimes P_\alpha) = H^1(A, \Omega_e \otimes P_\alpha) = 0$$

for any $e \geq 0$. Thus, we only need to consider whether $\hat{0} \in \hat{A}$ is in the cohomology support loci of Ω .

We have the following exact sequence,

$$\begin{aligned} 0 \rightarrow H^0(A, \mathcal{O}_A) \rightarrow H^0(A, \Omega_{e+1}) \rightarrow H^0(A, \Omega_e) \\ \xrightarrow{\cong} H^1(A, \mathcal{O}_A) \rightarrow H^1(A, \Omega_{e+1}) \rightarrow H^1(A, \Omega_e) \rightarrow 0, \end{aligned}$$

where the isomorphism is by the assumption that the extension of \mathcal{O}_A and Ω_e is non-splitting. Hence, $H^0(A, \Omega_{e+1}) \rightarrow H^0(A, \Omega_e)$ is zero and $H^1(A, \Omega_{e+1}) \rightarrow H^1(A, \Omega_e)$ is an isomorphism for any $e \geq 0$. Taking the inverse limit, we obtain that

$$H^0(A, \Omega) = H^0(A, \varprojlim \Omega_e) \cong \varprojlim H^0(A, \Omega_e) = 0,$$

and

$$H^1(A, \Omega) = H^1(A, \varprojlim \Omega_e) \cong \varprojlim H^1(A, \Omega_e) \cong H^1(A, \Omega_0) \cong k.$$

Thus, $V^0(\Omega) = \emptyset$ and $V^1(\Omega) = \{\hat{0}\}$. The chain of inclusions fails.

When A is supersingular, by [HST13, Lemma 4.12], we have that $F_*^e \omega_A \cong \Omega_{p^e-1}$. The nontrivial map $\Omega_{p^e-1} \rightarrow \Omega_0$ induces $F_*^e \omega_A \rightarrow \omega_A$, which is isomorphic to the trace map up to a scale. Hence, the inverse system $\{\Omega_{p^e-1}\}$ is a Cartier module and is the same as Example 3.1. \square

4. MAIN THEOREM

We will prove Theorem 1.5 in this section.

4.1. WIT versus limit of Kodaira vanishing.

Theorem 4.1. *Let A be an abelian variety of dimension g . Let $\{\Omega_e\}$ be an inverse system of coherent sheaves on A satisfying the Mittag-Leffler condition and let $\Omega = \varprojlim \Omega_e$. Let $\Lambda_e = R\hat{S}(D_A(\Omega_e))$ and $\Lambda = \varinjlim \Lambda_e$. The following are equivalent:*

- (1) *For any ample line bundle L on \hat{A} , $H^i(A, \Omega \otimes \hat{L}^\vee) = 0$ for any $i > 0$.*
- (2) *For any non-negative integer e and any ample line bundle L on \hat{A} , there exists an integer $m(e, L)$ such that for any $m \geq m(e, L)$, the natural map*

$$H^i(A, \Omega \otimes \widehat{mL}^\vee) \rightarrow H^i(A, \Omega_e \otimes \widehat{mL}^\vee)$$

is zero for any $i > 0$.

- (3) *$\mathcal{H}^i(\Lambda) = 0$ for $i \neq 0$.*

Proof. (1) \Rightarrow (2). Obvious.

(2) \Rightarrow (3). It is shown in [HP13, Theorem 3.1.1] that $\mathcal{H}^i(\Lambda) = 0$ when $i < -g$ or $i > 0$. Thus we may pick $j < 0$ as the smallest integer such that $\mathcal{H}^j(\Lambda) \neq 0$. Since $\mathcal{H}^j(\Lambda) = \varinjlim \mathcal{H}^j(\Lambda_e)$, we may fix $e > 0$ such that the image of $\mathcal{H}^j(\Lambda_e) \rightarrow \mathcal{H}^j(\Lambda)$ is non-zero. Let L be a sufficiently large multiple of a fixed ample line bundle on \hat{A} such that

- (i) $\mathcal{H}^j(\Lambda_e) \otimes L$ is globally generated,
- (ii) $H^i(\hat{A}, \mathcal{H}^l(\Lambda_e) \otimes L) = 0$ for $i > 0$ and $l \in [-g, 0]$, and
- (iii) $H^i(A, \Omega \otimes \hat{L}^\vee) \rightarrow H^i(A, \Omega_e \otimes \hat{L}^\vee)$ is zero for any $i \neq 0$.

Notice that (i) and (ii) can be achieved by Serre Vanishing and (iii) can be achieved by the hypothesis in condition (2).

Using Grothendieck's spectral sequence, we have

$$\begin{array}{ccc} E_{2,e}^{i,l} = R^i\Gamma(\mathcal{H}^l(\Lambda_e) \otimes L) & \Longrightarrow & R^{i+l}\Gamma(\Lambda_e \otimes L) \\ \downarrow & & \downarrow \\ E_2^{i,l} = R^i\Gamma(\mathcal{H}^l(\Lambda) \otimes L) & \Longrightarrow & R^{i+l}\Gamma(\Lambda \otimes L), \end{array}$$

where the vertical arrows are compatible by the functoriality of the spectral sequence. By our choice of j , we have that $E_2^{i,l} = 0$ for all $l < j$. By (ii), $E_{2,e}^{i,l} = 0$ for any $i \neq 0$. Hence, the spectral sequence degenerates to the following commutative diagram:

$$\begin{array}{ccc} R^0\Gamma(\mathcal{H}^j(\Lambda_e) \otimes L) & \xrightarrow{\simeq} & R^j\Gamma(\Lambda_e \otimes L) \\ \downarrow & & \downarrow \\ R^0\Gamma(\mathcal{H}^j(\Lambda) \otimes L) & \xrightarrow{\simeq} & R^j\Gamma(\Lambda \otimes L). \end{array}$$

By (i), the image of $R^0\Gamma(\mathcal{H}^j(\Lambda_e) \otimes L) \rightarrow R^0\Gamma(\mathcal{H}^j(\Lambda) \otimes L)$ is non-zero. Hence, the image of $R^j\Gamma(\Lambda_e \otimes L) \rightarrow R^j\Gamma(\Lambda \otimes L)$ is non-zero.

On the other hand, we have

$$\begin{aligned}
D_k(R^j\Gamma(\Lambda \otimes L)) &\cong D_k(\varinjlim R^j\Gamma(\Lambda_e \otimes L)) \\
&\cong \varprojlim D_k R^j\Gamma(\Lambda_e \otimes L) \\
&\cong \varprojlim D_k R^j\Gamma(R\hat{S}(D_A(\Omega_e)) \otimes L) \\
&\cong \varprojlim D_k R^j\Gamma(D_A(\Omega_e \otimes \hat{L}^\vee)) \\
&\cong \varprojlim R^{-j}\Gamma(D_A(D_A(\Omega_e \otimes \hat{L}^\vee))) \\
&\cong \varprojlim R^{-j}\Gamma(\Omega_e \otimes \hat{L}^\vee),
\end{aligned}$$

and similarly, $D_k(R^j\Gamma(\Lambda_e \otimes L)) \cong R^{-j}\Gamma(\Omega_e \otimes \hat{L}^\vee)$. Since the inverse system $\{\Omega_e\}$ satisfies the Mittag-Leffler condition, we have

$$H^i(A, \Omega \otimes \hat{L}^\vee) = H^i(A, \varprojlim \Omega_e \otimes \hat{L}^\vee) \cong \varprojlim H^i(A, \Omega_e \otimes \hat{L}^\vee)$$

for any i . Thus by (iii), $\varprojlim R^{-j}\Gamma(\Omega_e \otimes \hat{L}^\vee) \rightarrow R^{-j}\Gamma(\Omega_e \otimes \hat{L}^\vee)$ is zero. Hence, $D_k(R^j\Gamma(\Lambda \otimes L)) \rightarrow D_k(R^j\Gamma(\Lambda_e \otimes L))$ is zero, a contradiction.

(3) \Rightarrow (1). Recall that we have the following spectral sequence,

$$H^i(\hat{A}, \mathcal{H}^l(\Lambda) \otimes L) \Rightarrow R^{i+l}\Gamma(\hat{A}, \Lambda \otimes L).$$

Since $\mathcal{H}^i(\Lambda) = 0$ for any $i \neq 0$, the spectral sequence degenerates to

$$H^i(\hat{A}, \mathcal{H}^0(\Lambda) \otimes L) \cong R^i\Gamma(\hat{A}, \Lambda \otimes L).$$

If $i > 0$, then by the isomorphism shown in the previous step,

$$H^i(A, \Omega \otimes \hat{L}^\vee) \cong D_k(R^{-i}\Gamma(\hat{A}, \Lambda \otimes L)) \cong D_k(H^{-i}(\hat{A}, \mathcal{H}^0(\Lambda) \otimes L)) = 0.$$

□

4.2. WIT versus the supports of $R^i\hat{S}(\Omega)$.

Theorem 4.2. *Let $\{\Omega_e\}$ be an inverse system of coherent sheaves on a g -dimensional abelian variety satisfying the Mittag-Leffler condition and let $\Omega = \varprojlim \Omega_e$. Let $\Lambda_e = R\hat{S}(D_A(\Omega_e))$ and $\Lambda = \text{hocolim} \Lambda_e$. If $\mathcal{H}^j(\Lambda) = 0$ for any $j \neq 0$, then for any scheme-theoretic point P with $\dim P > i$, we have*

$$P \notin \text{Supp}(\text{im}(R^i\hat{S}(\Omega) \rightarrow R^i\hat{S}(\Omega_e)))$$

for any $e \geq 0$. Moreover, if $\mathcal{H}^0(\Lambda)$ is torsion-free, then for any scheme-theoretic point P with $\dim P \geq i$, we have

$$P \notin \text{Supp}(\text{im}(R^i\hat{S}(\Omega) \rightarrow R^i\hat{S}(\Omega_e)))$$

for any $e \geq 0$.

Proof. Fix a scheme-theoretic point $P \in \hat{A}$ such that $\dim P = d$. Since localization at P is exact, we have the following commutative diagram of spectral sequences:

$$\begin{array}{ccc}
\mathcal{E}xt^i(\mathcal{H}^j(\Lambda), \mathcal{O}_{\hat{A}})_P & \Longrightarrow & \mathcal{E}xt^{i-j}(\Lambda, \mathcal{O}_{\hat{A}})_P \\
\downarrow \phi_2^{i,-j} & & \downarrow \phi^{i-j} \\
\mathcal{E}xt^i(\mathcal{H}^j(\Lambda_e), \mathcal{O}_{\hat{A}})_P & \Longrightarrow & \mathcal{E}xt^{i-j}(\Lambda_e, \mathcal{O}_{\hat{A}})_P.
\end{array}$$

Since $\dim P = d$, by [Hartshorne77, III.6.8 and III.6.10A],

$$\mathcal{E}xt^i(\mathcal{H}^j(\Lambda_e), \mathcal{O}_{\hat{A}})_P \cong \text{Ext}_{\mathcal{O}_{\hat{A},P}}^i(\mathcal{H}^j(\Lambda_e)_P, \mathcal{O}_{\hat{A},P}) = 0,$$

when $i > g - d$. Let $l = i - j > g - d$ and $a = l - 1$. When $i \leq a$, we have $j = i - l < 0$, hence $\mathcal{E}xt^i(\mathcal{H}^j(\Lambda), \mathcal{O}_{\hat{A}})_P = 0$. When $i > a$, we have $i \geq l > g - d$, hence $\mathcal{E}xt^i(\mathcal{H}^j(\Lambda_e), \mathcal{O}_{\hat{A}})_P = 0$. We may apply Lemma 2.15 and obtain that the natural map $\mathcal{E}xt^l(\Lambda, \mathcal{O}_{\hat{A}})_P \rightarrow \mathcal{E}xt^l(\Lambda_e, \mathcal{O}_{\hat{A}})_P$ is zero when $l > g - d$.

It is easy to see that

$$\begin{aligned} \mathcal{E}xt^l(\Lambda_e, \mathcal{O}_{\hat{A}}) &\cong \mathcal{H}^{l-g}(D_{\hat{A}}(\Lambda_e)) \cong \mathcal{H}^{l-g}(D_{\hat{A}}(R\hat{S}(D_A(\Omega_e)))) \\ &\cong \mathcal{H}^{l-g}((-1_{\hat{A}})^* R\hat{S}(D_A(D_A(\Omega_e)))[g]) \cong \mathcal{H}^l((-1_{\hat{A}})^* R\hat{S}(\Omega_e)) \\ &\cong (-1_{\hat{A}})^* R^l \hat{S}(\Omega_e) \end{aligned}$$

and

$$\begin{aligned} \mathcal{E}xt^l(\Lambda, \mathcal{O}_{\hat{A}}) &\cong \mathcal{H}^{l-g}(D_{\hat{A}}(\Lambda)) \cong \mathcal{H}^{l-g}(D_{\hat{A}}(\text{hocolim} R\hat{S}(D_A(\Omega_e)))) \\ &\cong \mathcal{H}^{l-g}(\text{holim} D_{\hat{A}}(R\hat{S}(D_A(\Omega_e)))) \cong \mathcal{H}^l(\text{holim} (-1_{\hat{A}})^* R\hat{S}(\Omega_e)) \\ &\cong (-1_{\hat{A}})^* \mathcal{H}^l(\text{holim} R\hat{S}(\Omega_e)). \end{aligned}$$

Then for any d -dimensional point P ,

$$\mathcal{H}^l(\text{holim} R\hat{S}(\Omega_e))_P \rightarrow R^l \hat{S}(\Omega_e)_P$$

is zero for any $l > g - d$. Notice that $R^l \hat{S}(\Omega)_P \rightarrow R^l \hat{S}(\Omega_e)_P$ factors as

$$R^l \hat{S}(\Omega)_P = \mathcal{H}^l(R\hat{S}(\varprojlim \Omega_e))_P \rightarrow \mathcal{H}^l(\text{holim} R\hat{S}(\Omega_e))_P \rightarrow \mathcal{H}^l(R\hat{S}(\Omega_e))_P = R^l \hat{S}(\Omega_e)_P.$$

We conclude that $R^l \hat{S}(\Omega)_P \rightarrow R^l \hat{S}(\Omega_e)_P$ is zero when $l > g - \dim P$. The first part of the proposition follows from the exactness of localization at P .

If $\mathcal{H}^0(\Lambda)$ is torsion-free, we only need to check the case that $l = i - j = g - d$. Let $a = l$. When $i < a$, we have that $j = i - l < 0$, hence $\mathcal{E}xt^i(\mathcal{H}^j(\Lambda), \mathcal{O}_{\hat{A}})_P = 0$. When $i > a$, we have $i > l = g - d$, hence $\mathcal{E}xt^i(\mathcal{H}^j(\Lambda_e), \mathcal{O}_{\hat{A}})_P = 0$. To apply Lemma 2.15, we only need to check that

$$\mathcal{E}xt^{g-d}(\mathcal{H}^0(\Lambda), \mathcal{O}_{\hat{A}})_P \rightarrow \mathcal{E}xt^{g-d}(\mathcal{H}^0(\Lambda_e), \mathcal{O}_{\hat{A}})_P$$

is zero. Let \mathcal{T} be the torsion part of $\mathcal{H}^0(\Lambda_e)$ and $\mathcal{F} \cong \mathcal{H}^0(\Lambda_e)/\mathcal{T}$. We have the following exact sequence,

$$\mathcal{E}xt^{g-d}(\mathcal{F}, \mathcal{O}_{\hat{A}})_P \rightarrow \mathcal{E}xt^{g-d}(\mathcal{H}^0(\Lambda_e), \mathcal{O}_{\hat{A}})_P \rightarrow \mathcal{E}xt^{g-d}(\mathcal{T}, \mathcal{O}_{\hat{A}})_P \rightarrow \mathcal{E}xt^{g-d+1}(\mathcal{F}, \mathcal{O}_{\hat{A}})_P.$$

Since \mathcal{F} is torsion-free, by the argument in [PP08, Lemma 2.9], we have that

$$\mathcal{E}xt^{g-d}(\mathcal{F}, \mathcal{O}_{\hat{A}})_P = \mathcal{E}xt^{g-d+1}(\mathcal{F}, \mathcal{O}_{\hat{A}})_P = 0.$$

Hence

$$\mathcal{E}xt^{g-d}(\mathcal{H}^0(\Lambda_e), \mathcal{O}_{\hat{A}})_P \cong \mathcal{E}xt^{g-d}(\mathcal{T}, \mathcal{O}_{\hat{A}})_P.$$

We only need to show that

$$\mathcal{E}xt^{g-d}(\mathcal{H}^0(\Lambda), \mathcal{O}_{\hat{A}})_P \rightarrow \mathcal{E}xt^{g-d}(\mathcal{T}, \mathcal{O}_{\hat{A}})_P$$

is zero. Notice that this map is induced by $\mathcal{T} \rightarrow \mathcal{H}^0(\Lambda_e) \rightarrow \mathcal{H}^0(\Lambda)$ where \mathcal{T} is a torsion sheaf and $\mathcal{H}^0(\Lambda)$ is torsion-free. Thus $\mathcal{T} \rightarrow \mathcal{H}^0(\Lambda)$ is zero. Hence, $\mathcal{E}xt^{g-d}(\mathcal{H}^0(\Lambda), \mathcal{O}_{\hat{A}})_P \rightarrow \mathcal{E}xt^{g-d}(\mathcal{T}, \mathcal{O}_{\hat{A}})_P$ is zero. \square

4.3. The case when $\{R^i\hat{S}(\Omega_e)\}$ satisfies the Mittag-Leffler condition. In this section, we will consider the Mittag-Leffler condition on the Fourier-Mukai transform of the inverse system $\{\Omega_e\}$. We are able to recover Theorem 1.2 fully in this setting. However, we remind the reader that even when $\{\Omega_e\}$ is a Cartier module, the inverse system $\{R^i\hat{S}(\Omega_e)\}$ does not necessarily satisfy the Mittag-Leffler condition (see Example 3.3).

Proposition 4.3. *For any $0 \leq i \leq g$, if $\{R^i\hat{S}(\Omega_e)\}$ satisfies the Mittag-Leffler condition, then the support of $\text{im}(R^i\hat{S}(\Omega) \rightarrow R^i\hat{S}(\Omega_e))$ is closed for any $e \geq 0$.*

Proof. Since $\{\Omega_e\}$ satisfies the Mittag-Leffler condition, by Theorem 2.13, the natural map

$$R^i\hat{S}(\Omega) \rightarrow \varprojlim R^i\hat{S}(\Omega_e)$$

is surjective. Thus, we have

$$\text{im}(R^i\hat{S}(\Omega) \rightarrow R^i\hat{S}(\Omega_e)) = \text{im}(\varprojlim R^i\hat{S}(\Omega_e) \rightarrow R^i\hat{S}(\Omega_e)).$$

Since we assume $\{R^i\hat{S}(\Omega_e)\}$ satisfies the Mittag-Leffler condition, the image of

$$R^i\hat{S}(\Omega_d) \rightarrow R^i\hat{S}(\Omega_e)$$

stabilizes when d is sufficiently large. The stable image coincides with

$$\text{im}(\varprojlim R^i\hat{S}(\Omega_e) \rightarrow R^i\hat{S}(\Omega_e)).$$

Since $R^i\hat{S}(\Omega_d)$ and $R^i\hat{S}(\Omega_e)$ are both coherent, the proposition follows. \square

By the proposition above, we are able to talk about the codimension of the support of $\text{im}(R^i\hat{S}(\Omega) \rightarrow R^i\hat{S}(\Omega_e))$.

We are able to recover the missing implication in Theorem 1.5.

Theorem 4.4. *Let A be an abelian variety. Let $\{\Omega_e\}$ be an inverse system of coherent sheaves on A satisfying the Mittag-Leffler condition and let $\Omega = \varprojlim \Omega_e$. Let $\Lambda_e = R\hat{S}(D_A(\Omega_e))$ and $\Lambda = \varinjlim \Lambda_e$. Suppose that the inverse system $\{R^i\hat{S}(\Omega_e)\}$ satisfies the Mittag-Leffler condition for all i and*

$$\text{codim Supp}(\text{im}(R^i\hat{S}(\Omega) \rightarrow R^i\hat{S}(\Omega_e))) \geq i,$$

for any $0 \leq i \leq g$ and e sufficiently large. Then for any ample line bundle L on \hat{A} , we have $H^i(A, \Omega \otimes \hat{L}^\vee) = 0$ for any $i > 0$.

Proof. To simplify our notation, we denote

$$\text{im}_{i,e} = \text{im}(R^i\hat{S}(\Omega) \rightarrow R^i\hat{S}(\Omega_e)).$$

Let p and q be two non-negative integers satisfying $p + q > g$ and L be any ample line bundle on \hat{A} . The homomorphism

$$H^p(\hat{A}, R^q\hat{S}(\Omega) \otimes L^\vee) \rightarrow H^p(\hat{A}, R^q\hat{S}(\Omega_e) \otimes L^\vee)$$

factors through $H^p(\hat{A}, \text{im}_{q,e} \otimes L^\vee)$. Since we assume that

$$\text{codim Supp}(\text{im}_{q,e}) \geq q > g - p,$$

the cohomology

$$H^p(\hat{A}, \text{im}_{q,e} \otimes L^\vee) = 0.$$

Thus,

$$H^p(\hat{A}, R^q\hat{S}(\Omega) \otimes L^\vee) \rightarrow H^p(\hat{A}, R^q\hat{S}(\Omega_e) \otimes L^\vee)$$

is the zero map. Since $\{R^{q-1}\hat{S}(\Omega_e)\}$ satisfies the Mittag-Leffler condition, by Theorem 2.13, we have

$$R^q\hat{S}(\Omega) \cong \varprojlim R^q\hat{S}(\Omega_e).$$

Then by the Mittag-Leffler condition of $\{R^q\hat{S}(\Omega_e)\}$, we have the following isomorphism

$$H^p(\hat{A}, R^q\hat{S}(\Omega) \otimes L^\vee) \cong H^p(\hat{A}, \varprojlim R^q\hat{S}(\Omega_e) \otimes L^\vee) \cong \varprojlim H^p(\hat{A}, R^q\hat{S}(\Omega_e) \otimes L^\vee).$$

Combining with the zero map above, we obtain that the natural maps

$$\varprojlim H^p(\hat{A}, R^q\hat{S}(\Omega_e) \otimes L^\vee) \rightarrow H^p(\hat{A}, R^q\hat{S}(\Omega) \otimes L^\vee)$$

are all zero for any e sufficiently large. By the universal property of inverse limits, we conclude that

$$H^p(\hat{A}, R^q\hat{S}(\Omega) \otimes L^\vee) \cong \varprojlim H^p(\hat{A}, R^q\hat{S}(\Omega_e) \otimes L^\vee) = 0.$$

Consider the following spectral sequence

$$H^p(\hat{A}, R^q\hat{S}(\Omega) \otimes L^\vee) \Rightarrow H^{p+q}(\hat{A}, R\hat{S}(\Omega) \otimes L^\vee).$$

By the discussion above, $H^p(\hat{A}, R^q\hat{S}(\Omega) \otimes L^\vee) = 0$ if $p + q > g$. Hence,

$$H^l(\hat{A}, R\hat{S}(\Omega) \otimes L^\vee) = 0$$

for any $l > g$. We apply Theorem 2.13 with $T = \Gamma(\hat{S}(\bullet) \otimes L^\vee)$ and get

$$\varprojlim H^l(\hat{A}, R\hat{S}(\Omega_e) \otimes L^\vee) = H^l(\hat{A}, R\hat{S}(\Omega) \otimes L^\vee) = 0.$$

Since Ω_e and L^\vee are both coherent, we may apply Proposition 2.11 and obtain

$$\begin{aligned} H^l(\hat{A}, R\hat{S}(\Omega_e) \otimes L^\vee) &\cong H^l(A, \Omega_e \otimes RS(L^\vee)) \\ &\cong H^l(A, \Omega_e \otimes RS(D_{\hat{A}}(L)[-g])) \\ &\cong H^{l-g}(A, \Omega_e \otimes RS(D_{\hat{A}}(L))) \\ &\cong H^{l-g}(A, \Omega_e \otimes (-1_A)^* D_A(RS(L))[-g]) \\ &\cong H^{l-g}(A, \Omega_e \otimes (-1_A)^* \hat{L}^\vee), \end{aligned}$$

where the fourth isomorphism is by Proposition 2.8. Taking the inverse limit, we have

$$0 = \varprojlim H^l(\hat{A}, R\hat{S}(\Omega_e) \otimes L^\vee) \cong \varprojlim H^{l-g}(A, \Omega_e \otimes (-1_A)^* \hat{L}^\vee) \cong H^{l-g}(A, \Omega \otimes (-1_A)^* \hat{L}^\vee),$$

for any $l > g$. The theorem follows. \square

5. APPLICATIONS

Let A be an abelian variety of dimension g . We say that a sheaf \mathcal{F} on A satisfies IT_i for some $0 \leq i \leq g$ if $H^j(A, \mathcal{F} \otimes P_\alpha) = 0$ for any $j \neq i$ and $P_\alpha \in \text{Pic}^0(A)$. For example, if H is an ample line bundle, then H satisfies IT_0 .

We prove the following preservation of vanishing as a generalization of [PP08, Proposition 3.1 and Theorem 3.2].

Proposition 5.1. *Let $\{\Omega_e\}$ be a GV-inverse system of coherent sheaves. Let H be a locally free sheaf satisfying IT_0 . Then $\Omega \otimes H$ satisfies IT_0 .*

Proof. Consider any $\alpha \in \text{Pic}^0(A)$. Since H satisfies IT_0 , it follows that $R\hat{S}(H \otimes \alpha) = R^0\hat{S}(H \otimes \alpha)$ is a locally free sheaf on \hat{A} . Since $\{\Omega_e\}$ satisfies the Mittag-Leffler condition, we have

$$\begin{aligned} H^i(A, \Omega \otimes H \otimes \alpha) &\cong \varprojlim H^i(A, \Omega_e \otimes H \otimes \alpha) \\ &\cong \varprojlim D_k(H^i(A, \Omega_e \otimes H \otimes \alpha)) \\ &\cong D_k(\varprojlim H^{-i}(A, D_A(\Omega_e) \otimes (H \otimes \alpha)^\vee)) \\ &\cong D_k(H^{-i}(A, \varinjlim D_A(\Omega_e) \otimes (H \otimes \alpha)^\vee)) \\ &\cong D_k(\text{Ext}^{-i}(H \otimes \alpha, \varinjlim D_A(\Omega_e))) \\ &\cong D_k(\text{Ext}^{-i}(R^0\hat{S}(H \otimes \alpha), \mathcal{H}^0(\Lambda))) \\ &\cong D_k(H^{-i}(\hat{A}, \mathcal{H}^0(\Lambda) \otimes (R^0\hat{S}(H \otimes \alpha))^\vee)) \\ &\cong 0, \end{aligned}$$

when $i > 0$, where the sixth isomorphism is by Proposition 2.7. \square

Proposition 5.2. *Let $\{\Omega_e\}$ be a GV-inverse system of coherent sheaves. Let \mathcal{E} be a locally free coherent sheaf satisfying the Generic Vanishing conditions in the sense of [PP11]. Then $\{\Omega_e \otimes \mathcal{E}\}$ is a GV-inverse system.*

Proof. Let L be a sufficiently ample line bundle on \hat{A} . Since \mathcal{E} satisfies the Generic Vanishing conditions, by [PP08, Theorem 2.3(2)], $\mathcal{E} \otimes \hat{A}^\vee$ satisfies IT_0 . By Proposition 5.1, $\Omega \otimes \mathcal{E} \otimes \hat{A}^\vee$ also satisfies IT_0 . In particular, $H^i(A, \Omega \otimes \mathcal{E} \otimes \hat{A}^\vee) = 0$. The proposition follows. \square

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